CHAPITRE 4 : ASYMPTOTIC PROPERTIES OF MARKOV CHAINS

For simplicity, we assume that $X = \mathbb{R}^k$.

Définition 0.1 Let $P$ be a Markov kernel on $X \times \mathcal{B}(X)$. We say that $P$ is a coupling kernel of $(P, P)$ if and only if

\[ \forall (x, x') \in X^2, \forall A \in \mathcal{B}(X) \]
\[ P((x, x'), A \times X) = P(x, A) \]
\[ P((x, x'), X \times A) = P(x', A) \]

Exercice 1 Let $\xi, \xi'$ two probability measures on the measurable set $(X, \mathcal{B}(X))$. Let $\nu$ be a probability measure on $(X, \mathcal{B}(X))$ such that $d\xi = \nu \, dv$ and $d\xi' = \nu' \, dv$, that is $\varphi$ (resp. $\varphi'$) is the density of $\xi$ (resp. $\xi'$) wrt $\nu$. The total variation distance between $\xi$ and $\xi'$ is defined by:

\[ \|\xi - \xi'\|_{TV} = \int |\varphi(x) - \varphi'(x)| \, dv(x) = \sup_{|h| \leq 1} |\xi(h) - \xi'(h)| = \inf_{\mu \in \mathcal{C}(\xi, \xi')} \int 1(x \neq x') \mu(dx \, dx') \]

where $\mathcal{C}$ is the set of coupling distributions $\mu$ of $(\xi, \xi')$, that is, distributions $\mu$ on $(X^2, \mathcal{B}(X) \otimes \mathcal{B}(X))$ such that the following marginal conditions are satisfied: for all $A \in \mathcal{B}(X)$,

\[ \mu(A \times X) = \xi(A) \]
\[ \mu(X \times A) = \xi'(A) \]

1. Show the different equalities that appear in the definition of the Total Variation distance.

Exercice 2 Let $P$ be a Markov kernel on $X \times \mathcal{B}(X)$ such that

(i) There exist a measurable function $V : X \to [1, \infty)$, $\lambda \in (0, 1)$ and $b \in \mathbb{R}$ such that for all $x \in X$,

\[ PV(x) \leq \lambda V(x) + b \]

(ii) There exist $M > 0$, $\epsilon > 0$ and a probability measure $\nu$ on $(X, \mathcal{B}(X))$ such that for all $x \in C_M \overset{\text{def}}{=} \{x \in X : V(x) \leq M\}$,

\[ P(x, \cdot) \geq \epsilon \nu(\cdot) \]

(iii) $\lambda \overset{\text{def}}{=} \lambda + \frac{2b}{1 + M} < 1$.

In what follows, we use the notation $\hat{x} = (x, x')$ and $\hat{C}_M = C_M \times C_M$. Writing for all $x \in C_M$, $Q(x, dy) = \frac{P(x, dy) - \nu(dy)}{1 - \epsilon}$, we define the Markov kernel $P$ on $X^2 \times \mathcal{B}(X \otimes X)$ by

\[ P((x, x'); dy, dy') = 1_{\hat{C}_M}(\hat{x}) \left[ \nu(dy) \delta_y(dy') + (1 - \epsilon)Q(x, dy)Q(x', dy') \right] + 1_{\hat{C}_M}(\hat{x})P(x, dy)P(x', dy') \]
1. Show that $\bar{P}$ is a kernel coupling of $(P, P)$.

2. Define $d(x, x') = 1(x \neq x')$. Show that $\bar{P}d(x, x') \leq (1 - \varepsilon)d(x, x')$ for all $\bar{x} = (x, x') \in C_M$.

3. Define $\bar{V}(x, x') = \frac{V(x) + V(x')}{2}$. Show that $P\bar{V}(x, x') \leq \lambda\bar{V}(x, x') \leq C_M$ for all $\bar{x} = (x, x') \notin C_M$.

4. Deduce that there exists $\delta, \rho \in (0, 1)$ such that defining $W = V^{1 - \delta}$, we have

$$P W \leq \rho W$$

5. Show that for all $x, x' \in X$ and all $n \in \mathbb{N}$,

$$\|P^n(x, \cdot) - P^n(x', \cdot)\|_{TV} \leq 2\rho^n W(x, x')$$

6. Deduce that $P$ admits an invariant probability measure $\pi$. We admit that the set of probability measures on $(X, B(X))$ equipped with the Total Variation distance is a complete space.

Exercice 3 In this exercise, we consider the same assumptions as in Exercise 2. For any real-valued function $h$ on $X$ such that $\pi(|h|) < \infty$, we say that a real-valued function $\hat{h}$ on $X$ solves the Poisson equation associated to $f$ if for all $x \in X$,

$$\hat{h}(x) - P\hat{h}(x) = h(x) - \pi(h)$$

provided that $P\hat{h}(x)$ is well-defined for all $x \in X$. Define $S_n(h) = \sum_{k=0}^{n-1} \{h(X_k) - \pi(h)\}$.

1. Show that

$$S_n(h) = M_n(h) + \hat{h}(X_0) - \hat{h}(X_n)$$

where $M_n(h) = \sum_{k=1}^{n} \{\hat{h}(X_k) - P\hat{h}(X_{k-1})\}$.

2. Show that $\{M_n(h)\}_{n \geq 0}$ is an $(\mathcal{F}_n)_{n \geq 0}$-martingale, where we have set $\mathcal{F}_n = \sigma(X_0, \ldots, X_n)$.

3. Show that for all bounded function $h$, the function $\hat{h}(x) = \sum_{k=0}^{n} \{P^k(h)(x) - \pi(h)\}$ solves the Poisson equation.