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Simulation and inference of stochastic differential equations

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Chapter 1

Discretization of stochastic differential equations

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1.1 Gentle reminders: Gaussian vectors, Brownian motion, Stochastic differential equations

1.1.1 Gaussian random vectors

Definition 1.1. A random variable \(X \in \mathbb{R}^n\) is a Gaussian vector if and only if, for all \(a \in \mathbb{R}^n\), the random variable \(\langle a ; X \rangle\) is a Gaussian random variable.

For all random variable \(X \in \mathbb{R}^n\), \(X \sim \mathcal{N}(\mu, \Sigma)\) means that \(X\) is a Gaussian vector with mean \(\mathbb{E}[X] = \mu \in \mathbb{R}^n\) and covariance matrix \(\mathbb{V}[X] = \Sigma \in \mathbb{R}^{n \times n}\). The characteristic function of \(X\) is given, for all \(t \in \mathbb{R}^n\), by:

\[
\mathbb{E}\left[e^{i\langle t ; X \rangle}\right] = e^{i\langle t ; \mu \rangle - \frac{1}{2} t' \Sigma t}.
\]

Therefore, the law of a Gaussian vector is uniquely defined by its mean vector and its covariance matrix. If the covariance matrix \(\Sigma\) is nonsingular, then the law of \(X\) has a probability density with respect to the Lebesgue measure on \(\mathbb{R}^n\) given by:
Proposition 1.2. Let \( X \in \mathbb{R}^n \) be a Gaussian vector. Let \( \{i_1, \ldots, i_d\} \) be a subset of \( \{1, \ldots, n\} \), \( d \geq 1 \). If for all \( 1 \leq k \neq j \leq d \), \( \text{Cov}(X_{i_k}, X_{i_j}) = 0 \), then \( (X_{i_1}, \ldots, X_{i_d}) \) are independent.

Proof. The random vector \( (X_{i_1}, \ldots, X_{i_d})' \) is a Gaussian vector with mean \( (\mathbb{E}[X_{i_1}], \ldots, \mathbb{E}[X_{i_d}])' \) and diagonal covariance matrix \( \text{diag}(\mathbb{E}[X_{i_1}^2], \ldots, \mathbb{E}[X_{i_d}^2]) \). Consider \( (\xi_{i_1}, \ldots, \xi_{i_d}) \) i.i.d. random variables with distribution \( \mathcal{N}(0, 1) \) and define, for all \( 1 \leq j \leq d \),

\[
Z_{i_j} = \mathbb{E}[\xi_{i_j}] + \sqrt{\mathbb{E}[X_{i_j}^2]} \xi_{i_j}.
\]

Then, the random vector \( (Z_{i_1}, \ldots, Z_{i_d})' \) is a Gaussian vector with the same mean and the same covariance matrix as \( (X_{i_1}, \ldots, X_{i_d})' \). The two vectors have therefore the same characteristic function and the same law and \( (X_{i_1}, \ldots, X_{i_d}) \) are independent as \( (\xi_{i_1}, \ldots, \xi_{i_d}) \) are independent.

Theorem 1.3 (Cochran). Let \( X \sim \mathcal{N}(0, I_n) \) be a Gaussian vector in \( \mathbb{R}^n \), \( F \) be a vector subspace of \( \mathbb{R}^n \) and \( F^\perp \) its orthogonal. Denote by \( \pi_F(X) \) (resp. \( \pi_{F^\perp}(X) \)) the orthogonal projection of \( X \) on \( F \) (resp. on \( F^\perp \)). Then, \( \pi_F(X) \) and \( \pi_{F^\perp}(X) \) are independent, \( \|\pi_F(X)\|^2 \sim \chi^2(p) \) and \( \|\pi_{F^\perp}(X)\|^2 \sim \chi^2(n-p) \), where \( p \) is the dimension of \( F \).

Proof. Let \( (u_1, \ldots, u_n) \) be an orthonormal basis of \( \mathbb{R}^n \) where \( (u_1, \ldots, u_p) \) is an orthonormal basis of \( F \) and \( (u_{p+1}, \ldots, u_n) \) and orthonormal basis of \( F^\perp \). Consider the matrix \( U \in \mathbb{R}^{n \times n} \) such that for all \( 1 \leq i \leq n \), the \( i \)-th column of \( U \) is \( u_i \) and \( U_{(p)} \) (resp. \( U_{(n-p)} \)) the matrix made of the first \( p \) (resp. last \( n-p \)) columns of \( U \). Note that

\[
\pi_F(X) = \sum_{i=1}^p \langle X, u_i \rangle u_i,
\]

which can be written \( \pi_F(X) = U_{(p)}U'_{(p)}X \). Similarly, \( \pi_{F^\perp}(X) = U_{(n-p)}U'_{(n-p)}X \).

Therefore,

\[
\begin{pmatrix}
\pi_F(X) \\
\pi_{F^\perp}(X)
\end{pmatrix} =
\begin{pmatrix}
U_{(p)}U'_{(p)} \\
U_{(n-p)}U'_{(n-p)}
\end{pmatrix} X
\]

is a centered Gaussian vector with covariance matrix given by

\[
\begin{pmatrix}
U_{(p)}U'_{(p)} & 0 \\
0 & U_{(n-p)}U'_{(n-p)}
\end{pmatrix}.
\]

By Proposition 1.2, \( \pi_F(X) \) and \( \pi_{F^\perp}(X) \) are independent. On the other hand,

\[
\|\pi_F(X)\|^2 = \sum_{i=1}^p \langle X, u_i \rangle^2 \quad \text{and} \quad \|\pi_{F^\perp}(X)\|^2 = \sum_{i=p+1}^n \langle X, u_i \rangle^2.
\]

The random vector \( (\langle X, u_i \rangle)_{1 \leq i \leq n} \) is given by \( U'X \); it is a Gaussian random vector with mean 0 and covariance matrix \( I_n \). The random variables \( (\langle X, u_i \rangle)_{1 \leq i \leq n} \) are therefore i.i.d. with distribution \( \mathcal{N}(0, 1) \), which concludes the proof.
1.1 Gentle reminders: Gaussian vectors, Brownian motion, Stochastic differential equations

### 1.1.2 Brownian motion

**Definition 1.4.** A continuous time process \((W_t)_{t>0}\) is a Brownian motion started at 0 if and only if:

1. \(W_0 = 0\).
2. \((W_t)_{t>0}\) is a Gaussian process.
3. For all \((s,t) \in \mathbb{R}_+^2\), \(W_t - W_s \sim \mathcal{N}(0, t-s)\).
4. For all \((s,t) \in \mathbb{R}_+^2\), \(s \leq t\), \(W_t - W_s\) is independent of \(\sigma((W_u)_{0 \leq u \leq s})\).
5. The trajectory \(t \mapsto W_t\) is continuous.

When the trajectory \(t \mapsto W_t\) is not assumed to be continuous, it can be shown that assumptions i) to iii) imply that it is almost surely continuous.

**Proposition 1.5** A Gaussian process \((W_t)_{t>0}\) with continuous trajectories and started at 0 is a Brownian motion if and only if the following properties hold:

- For all \(t \geq 0\), \(\mathbb{E}[W_t] = 0\).
- For all \((s,t) \in \mathbb{R}_+^2\), \(\mathbb{E}[W_sW_t] = \min(s,t) = s \wedge t\).

**Proof.** Assume that \((W_t)_{t>0}\) is a Brownian motion.

- \(\forall t \geq 0\), \(\mathbb{E}[W_t] = \mathbb{E}[W_t - W_0] = 0\) since \(W_t - W_0 \sim \mathcal{N}(0, t)\).
- For all \((s,t) \in \mathbb{R}_+^2\) such that \(s \leq t\),
  \[
  \mathbb{E}[W_sW_t] = \mathbb{E}[W_s(W_t - W_s) + W_sW_s] = \mathbb{E}[W_s^2] + \mathbb{E}[W_s(W_t - W_s)] = \mathbb{E}[(W_t - W_0)^2] + \mathbb{E}[W_s] \mathbb{E}[W_t - W_s] = s + 0 = s
  \]

Assume now that \(\mathbb{E}[W_t] = 0\) and that for all \((s,t) \in \mathbb{R}_+^2\), \(\mathbb{E}[W_sW_t] = s \wedge t\).

- Let \((s,t) \in \mathbb{R}_+^2\) with \(s < t\). To prove that \(W_t - W_s\) is independent of \(\sigma((W_u)_{0 \leq u \leq s})\), it is enough to prove that for all \(0 \leq u \leq s\), \(\text{Cov}(W_t - W_s, W_u) = 0\). Note that,
  \[
  \text{Cov}(W_t - W_s, W_u) = \mathbb{E}[W_uW_t] - \mathbb{E}[W_uW_s] = u \wedge t - u \wedge s = u - u = 0.
  \]

For all \((s,t) \in \mathbb{R}_+^2\), \(s \leq t\), \(W_t - W_s\) is Gaussian and centerer. In addition,

- \(\forall W_t - W_s = e)((W_t - W_s)^2\) \(= \mathbb{E}[W^2] + \mathbb{E}[W^2] - 2\mathbb{E}[W_sW_t] = t+s-2s \wedge t = t - s\).

**Corollary 1.6** Let \((W_t)_{t>0}\) be a Brownian motion. Then, the following processes are also Brownian motions:

- \((W_{t+t_0} - W_{t_0})_{t>0}\) for all \(t_0 \in \mathbb{R}_+\).
- \((tW_t)_{t>0}\).
- \((\alpha W_t/\alpha^2)_{t>0}\) for all \(\alpha > 0\).

**Proof.** See exercises.
Proposition 1.7 Let \((W_t)_{t \geq 0}\) be a Brownian motion. Then,
- \(\limsup_{t \to +\infty} \frac{W_t}{\sqrt{t}} = +\infty\) (and then \(\limsup_{t \to +\infty} W_t = +\infty\)) almost surely.
- The Brownian motion takes almost surely each real value infinitely many often.

### 1.1.3 Stochastic differential equations

#### 1.1.3.1 Construction

In the case of Riemann integrals, for all \(T > 0\) and all continuous function \(f : [0, T] \to \mathbb{R}\), define

\[
I_{n,T}(f) = \sum_{i=0}^{n} f(t^n_i) (t^n_{i+1} - t^n_i),
\]

where \((t^n_i)_{0 \leq i \leq n+1}\) is a subdivision of \([0, T]\), \(t^n_0 = 0 < t^n_1 < \ldots < t^n_{n+1} = T\). If \(\sup_{0 \leq i \leq n+1} (t^n_{i+1} - t^n_i) \to 0\), then \(I_{n,T}(f)\) converges as \(n\) grows to infinity to a quantity denoted \(\int_0^T f(u) \, du\).

Following this construction, consider a piecewise constant process on \([0, T]\), defined, for all \(t \in [0, T]\) by

\[
X_t = \sum_{i=0}^{n} X^n_i 1_{[t^n_i, t^n_{i+1})}(t),
\]

where \(X^n_i\) is a \(\mathcal{F}^n\)-measurable random variable and \(t^n_0 = 0 < t^n_1 < \ldots < t^n_{n+1} = T\). Then, define

\[
\int_0^T X_t \, dW_s = \sum_{i=0}^{n} X^n_i (W^n_{i+1} - W^n_i).
\]

If \((X_t)_{0 \leq t \leq T}\) is a bounded continuous process, the stochastic integral \(\int_0^T X_t \, dW_s\) is constructed as follows.

- For all \(n \geq 1\), define

\[
X^n_t = \sum_{k=0}^{n+1} X^n_k 1_{[kT/n, (k+1)T/n)}(t)
\]

and the associated stochastic integral \(M^n_T = \int_0^T X^n_t \, dW_s\).

- Then, it may be shown that \((M^n_T)_{n \geq 0}\) converges in \(\mathcal{L}^2\) to a random variable denoted \(M_T := \mathbb{E}[(M^n_T - M_T)^2] \to_{n \to \infty} 0\). This random variable is written \(M_T = \int_0^T X_t \, dW_s\).

In this chapter, the objective is to sample solutions to stochastic differential equations (SDE) of the form:

\[
dX_t = \alpha_\theta(X_t) \, ds + \sigma_\theta(X_t) \, dW_s, \tag{1.1}
\]

with \(\theta \in \mathbb{R}^d\) an unknown parameter to be estimated and

- \(\alpha_\theta : \mathbb{R} \to \mathbb{R}\) and \(\sigma_\theta : \mathbb{R} \to \mathbb{R}\) are two continuous functions.
- \((W_t)_{t \geq 0}\) is a Brownian motion associated with its filtration \(\mathcal{F}_t = \sigma((W_s)_{0 < s \leq t})\).

The process \((X_t)\) is said to be a strong solution to (1.1) if and only if, almost surely, for all \(t \geq 0\),

\[
X_t = X_0 + \int_0^t \alpha_\theta(X_s) \, ds + \int_0^t \sigma_\theta(X_s) \, dW_s.
\]

#### Example 1.8 (Movement ecology)

The process \((X_t)_{t \geq 0}\) is the 2-dimensional position of an individual:
1.2 Simulation of the Brownian motion

\[ dX_t = \nabla_A A_\theta(X_t) \, ds + \sigma \, dW_t , \]

where \( A_\theta : \mathbb{R}^2 \to \mathbb{R} \) is a potential function. In this framework, the movement is supposed to reflect the attractiveness of the environment, which is modeled using a real valued potential function defined on \( \mathbb{R}^2 \). The position may be observed only indirectly as follows:

\[ Y_k = X_k + \varepsilon_k , \]

where the \( (\varepsilon_k)_{0 \leq k \leq n} \) are i.i.d. \( \mathcal{N}(0, \eta^2 I_2) \).

1.2 Simulation of the Brownian motion

1.2.1 Simulation of a skeleton

Assume that a time horizon \( T > 0 \) and \( n \) time steps \( (t_1, \ldots, t_n) \) are defined such as \( 0 < t_1 < \ldots < t_n < T \). Consider the following algorithm to sample \( (W_{t_1}, \ldots, W_{t_n}) \).

- Sample \( (\varepsilon_1, \ldots, \varepsilon_n) \) i.i.d. with \( \varepsilon_1 \sim \mathcal{N}(0, t_1) \) and \( \varepsilon_i \sim \mathcal{N}(0, t_i - t_{i-1}) \) for \( 1 < i \leq n \).

Define \( X_0 = \varepsilon_1 \) and for \( i > 1 \), \( X_i = X_{i-1} + \varepsilon_i \).

Then, choosing, \( X_0 = 0 \) yields \( (X_1, \ldots, X_n) \overset{d}{=} (W_{t_1}, \ldots, W_{t_n}) \).

**Proof.** Note that,

\[ (X_{i_1}, X_{i_2} - X_{i_1}, \ldots, X_{n} - X_{n-1}) = (\varepsilon_{i_1}, \ldots, \varepsilon_n) \overset{d}{=} (W_{t_{i_1}}, W_{t_{i_2}} - W_{t_{i_1}}, \ldots, W_{t_n} - W_{t_{n-1}}) \]

and, since \( X_0 = W_0 = 0 \),

\[ (X_0, X_1, X_2 - X_1, \ldots, X_n - X_{n-1}) \overset{d}{=} (W_0, W_{t_1}, W_{t_2} - W_{t_1}, \ldots, W_{t_n} - W_{t_{n-1}}) . \]

By linear transformation, it yields

\[ (X_0, X_1, \ldots, X_n) \overset{d}{=} (W_0, W_{t_1}, \ldots, W_{t_n}) . \]

1.2.2 Completion of a Brownian trajectory

Assume that \( (W_{t_1}, \ldots, W_{t_n}) \) is the same distribution as a Brownian motion at times \( (t_1, \ldots, t_n) \). Conditionally to these random variables, the aim of this section is to sample the Brownian motion at other time steps.

**Lemma 1.9** Assume that \( (X, Y, Z) \) is a centered Gaussian random vector. The, the conditional distribution of \( Y \) given \( (X, Z) \) is Gaussian with mean \( \pi_{(X,Z)}(Y) \) and variance \( ||Y - \pi_{(X,Z)}(Y)||^2 \) where \( \pi_{X,Z} \) is the orthogonal projection on the vector space generated by \( (X, Z) \) for the scalar product \( \langle U, V \rangle \rightarrow \mathbb{E}[UV] \).

**Proof.** Soon.

The objective is now to sample \( W_u \) conditionally to \( (W_{t_1}, \ldots, W_{t_n}) \) where \( u \in (t_k, t_{k+1}) \), \( k \in \{1, \ldots, n-1\} \).

This law is the law of \( W_u \) given \( (W_{t_k}, W_{t_{k+1}}) \). As \( (W_{t_k}, W_{u}, W_{t_{k+1}}) \) is a centered Gaussian random vector, it is enough to apply Lemma 1.9, i.e. to compute \( \pi_{(W_{t_k}, W_{t_{k+1}})}(W_u) \) and \( ||W_u - P_{(W_{t_k}, W_{t_{k+1}})}(W_u)||^2 \). As \( (W_{t_k}, W_{t_{k+1}} - W_{t_k}) \) is an orthogonal basis of \( \text{Span}(W_{t_k}, W_{t_{k+1}}) \), for \( \langle \cdot, \cdot \rangle \).

Then,
\[ \pi(W_{k}, W_{k+1}) (W_u) = \begin{pmatrix} W_{k+1} - W_k \\ W_{k+1} \end{pmatrix} + \left( \begin{pmatrix} W_{k+1} - W_k \\ W_{k+1} - W_k \end{pmatrix} \right)^{\top} \begin{pmatrix} W_{k+1} - W_k & W_{k+1} - W_k \end{pmatrix} \]

\[ = \langle W_u, W_{k+1} \rangle W_k + \frac{\langle W_u, W_{k+1} - W_k \rangle}{W_{k+1} - W_k} \left( W_{k+1} - W_k \right). \]

Note that,

\[ \langle W_k, W_k \rangle = \mathbb{E}[W_k^2] = t_k \]

and

\[ \langle W_{k+1} - W_k, W_{k+1} - W_k \rangle = \mathbb{E}[\langle W_{k+1} - W_k \rangle^2] = t_{k+1} - t_k. \]

In addition,

\[ \langle W_u, W_k \rangle = \mathbb{E}[W_u W_k] = u \land t_k = t_k \]

and

\[ \langle W_u, W_{k+1} - W_k \rangle = \langle W_u, W_{k+1} \rangle - \langle W_u, W_k \rangle = \mathbb{E}[W_u W_{k+1}] - \mathbb{E}[W_u W_k] = w \land t_{k+1} - w \land t_k = u - t_k \]

Conditionally to \( (W_k, W_{k+1}) \) the mean of \( W_u \) is \( \mathbb{E}[W_u | W_k, W_{k+1}] = W_k + \frac{u - t_k}{t_{k+1} - t_k} (W_{k+1} - W_k) \), i.e.,

\[ \mathbb{E}[W_u | W_k, W_{k+1}] = \frac{t_{k+1} - u}{t_{k+1} - t_k} W_k + \frac{u - t_k}{t_{k+1} - t_k} W_{k+1}. \]

The conditional variance is

\[ \mathbb{E}[(W_u - \mathbb{E}(W_u | W_k, W_{k+1}))^2] = \left( \frac{t_{k+1} - u}{t_{k+1} - t_k} \right)^2 \mathbb{E}[(W_k - W_u)^2] + \left( \frac{u - t_k}{t_{k+1} - t_k} \right)^2 \mathbb{E}[(W_{k+1} - W_u)^2], \]

\[ = \frac{(t_{k+1} - u)^2 (u - t_k)}{(t_{k+1} - t_k)^2} + \frac{(u - t_k)^2 (t_{k+1} - u)}{(t_{k+1} - t_k)^2}, \]

\[ = \frac{(t_{k+1} - u) (u - t_k)}{t_{k+1} - t_k}. \]

### 1.3 Discretization of SDE

In this section, \( (X_t)_{0 \leq t \leq T} \) is solution to the following SDE:

\[ dX_t = b(X_t) dt + \sigma(X_t) dW_t, \quad (1.2) \]

where \( b : \mathbb{R} \rightarrow \mathbb{R} \) and \( \sigma : \mathbb{R} \rightarrow \mathbb{R} \) are continuous. Assume that \( \sigma \) and \( b \) are lipschitz: there exists \( K \in \mathbb{R}_+ \) such that for all \( (x, y) \in \mathbb{R}^2 \),

\[ |b(x) - b(y)| + |\sigma(x) - \sigma(y)| \leq K|x - y|, \]

then there exists a unique process \( (X_t)_{0 \leq t \leq T} \) such that, almost surely, for all \( 0 \leq t \leq T \),

\[ X_t - X_0 = \int_0^t b(X_s) ds + \int_0^t \sigma(X_s) dW_s. \]
1.3 Discretization of SDE

1.3.1 Euler-Maruyama scheme

Consider the evenly spaced partition of $[0,T]$ given by $(t^n_k = kT/N)_{0 \leq k \leq n}$. To obtain a sample approximatively distributed as $(X^n_0, \ldots, X^n_n)$, the drift and diffusion of the SDE are assumed to be fixed on each interval $(t^n_k, t^n_{k+1})$ for $0 \leq k \leq n-1$. Then, the approximate samples are defined as $\tilde{X}_n = X_0$ and for all $k \in \{0, \ldots, n-1\}$,

$$\tilde{X}^n_{k+1} = \tilde{X}^n_k + \frac{T}{n} b(\tilde{X}^n_k) + \sigma(\tilde{X}^n_k) \sqrt{\frac{T}{n}} \epsilon_{k+1},$$

where $(\epsilon_k)_{1 \leq k \leq n}$ are i.i.d. with distribution $\mathcal{N}(0,1)$. A continuous approximation is given by $\overline{X}_n = X_0$ and for all $k \in \{0, \ldots, n-1\}$, and all $t \in [t^n_k, t^n_{k+1})$,

$$\overline{X}_t - \overline{X}^n_t = b(\overline{X}^n_t)(t - t^n_k) + \sigma(\overline{X}^n_t)(W_t - W^n_{t^n_k}).$$

1.3.2 Approximation error for a Brownian motion

In the following, write for all $0 \leq k \leq n-1$ and all $t \in [t^n_k, t^n_{k+1})$, $\xi = t^n_k$.

1.3.2.1 Lower bound

The objective here is to quantify, for $p \geq 2$, $\left\| \sup_{t \in [0,T]} |W_t - \overline{X}_t| \right\|_p$. Using elementary algebra yields:

$$\left\| \sup_{t \in [0,T]} |W_t - \overline{X}_t| \right\|_p = \left\| \max_{k \in \{1, \ldots, n\}} \sup_{t \in [t^n_k, t^n_{k+1})} |W_t - W^n_{t^n_k}| \right\|_p,$$

$$= \sqrt{\frac{T}{n}} \left\| \max_{k \in \{1, \ldots, n\}} \sup_{t \in [t^n_k, t^n_{k+1})} \frac{n}{T} |W_t - W^n_{t^n_k}| \right\|_p,$$

$$= \sqrt{\frac{T}{n}} \left\| \max_{k \in \{1, \ldots, n\}} \sup_{t \in [t^n_k, t^n_{k+1})} \sqrt{n/T} |W_t - W^n_{t^n_k}| \right\|_p,$$

$$= \sqrt{\frac{T}{n}} \left\| \max_{k \in \{1, \ldots, n\}} \sup_{t \in [t^n_k, t^n_{k+1})} \sqrt{n/T} |W_t - W^n_{t^n_k}| \right\|_p,$$

$$= \sqrt{\frac{T}{n}} \left\| \max_{k \in \{1, \ldots, n\}} \epsilon_k \right\|_p,$$

where $\epsilon_k = \sup_{t \in [t^n_k, t^n_{k+1})} |W_t - W^n_{t^n_k}|$ and where we used that $(\sqrt{n/T}W_t/n)_{t \in [0,T]}$ is a Brownian motion. For all $k \in \{1, \ldots, n\}$, $\epsilon_k \geq |W_k - W^n_{t^n_k}| \geq 0$, so that
where \( c_p > 0 \) does not depend on \( n \) and where we used that \( (|W_k - W_{k-1}|)_{1 \leq k \leq n} \) are i.i.d. with \( W_k - W_{k-1} \sim \mathcal{N}(0, 1) \). The last inequality is left to the reader.

### 1.3.2.2 Upper bound

To obtain an upper bound, write

\[
\| \sup_{t \in [0,T]} |W_t - W_0| \|_p \geq \sqrt{\frac{T}{n}} \| \sup_{k \in \{1, \ldots, n\}} |W_k - W_{k-1}| \|_p ,
\]

\[
\geq \sqrt{\frac{T}{n}} \| \sup_{k \in \{1, \ldots, n\}} |W_k - W_{k-1}| \|_p ,
\]

\[
\geq \sqrt{\frac{T}{n}} \sqrt{\| \sup_{k \in \{1, \ldots, n\}} |W_k - W_{k-1}|^2 \|_2} ,
\]

\[
\geq \sqrt{\frac{T}{n}} c_p \sqrt{\log n} ,
\]

where \( c_p > 0 \) does not depend on \( n \) and where we used that \( (|W_k - W_{k-1}|)_{1 \leq k \leq n} \) are i.i.d. with \( W_k - W_{k-1} \sim \mathcal{N}(0, 1) \). The last inequality is left to the reader.

### 1.3.2.2 Upper bound

To obtain an upper bound, write

\[
\| \sup_{t \in [0,T]} |W_t - W_0| \|_p = \sqrt{\frac{T}{n}} \left( \max_{t \in [0,T]} \sup_{k \in \{1, \ldots, n\}} |W_t - W_{k-1}|^2 \right)^{\frac{1}{2}} ,
\]

where for all \( 1 \leq k \leq n \), \( \sup_{t \in [k-1,k]} |W_t - W_{k-1}|^2 \overset{\text{law}}{=} \sup_{t \in [0,1]} |W_t|^2 = \varepsilon \). Then, it can be proved that if for some \( \lambda > 0 \), \( \mathbb{E}[e^{\lambda \varepsilon}] < +\infty \),

\[
\| \max_{k \in \{1, \ldots, n\}} \sup_{t \in [k-1,k]} |W_t - W_{k-1}|^2 \|_2 \lesssim c_{p,\lambda} \log(n + 1) ,
\]

where \( c_{p,\lambda} \) does not depend on \( n \). For all \( \lambda > 0 \),

\[
\mathbb{E}[e^{\lambda \varepsilon}] = \mathbb{E}[e^{\lambda \sup_{t \in [0,1]} |W_t|^2}] ,
\]

\[
= \mathbb{E}[e^{\lambda \max(\sup_{t \in [0,1]} |W_t|, \sup_{t \in [0,1]} (-W_t))^2}] ,
\]

\[
\leq \mathbb{E}[e^{\lambda \sup_{t \in [0,1]} |W_t|^2} + e^{\lambda \sup_{t \in [0,1]} (-W_t)^2}] ,
\]

\[
\leq 2 \mathbb{E}[e^{\lambda (\sup_{t \in [0,1]} |W_t|^2 + (-W_t)^2)}] ,
\]

as \( (-W_t)_{0 \leq t \leq 1} \) has the same law as \( (W_t)_{0 \leq t \leq 1} \). By the reflection principle (see exercises), \( \sup_{t \in [0,1]} W_t = |W_1| \). Then, as \( W_1 \sim \mathcal{N}(0, 1) \), if \( \lambda \in (0, \frac{1}{2}) \),

\[
\mathbb{E}[e^{\lambda \varepsilon}] \leq 2 \mathbb{E}[e^{\lambda |W_1|^2}] \leq 2 \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{\frac{\lambda^2 x^2}{2}} dx \leq 2 \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-\frac{x^2}{2}} (1 - 2\lambda)x^2 dx ,
\]

\[
\leq \frac{2}{\sqrt{1 - \lambda}} \frac{1}{\sqrt{2\pi}} \sqrt{1 - 2\lambda} \int_{\mathbb{R}} e^{\frac{-x^2}{2(1 - 2\lambda)^2}} dx ,
\]

\[
\leq \frac{2}{\sqrt{1 - \lambda}}.
\]

Then, \( \mathbb{E}[e^{\lambda \varepsilon}] < +\infty \) which concludes the proof.
1.3 Discretization of SDE

1.3.3 $L_p$-mean error for general SDE

**Lemma 1.10 (Gronwald)** Let $f : \mathbb{R}_+ \to \mathbb{R}_+$ be a continuous and locally bounded function and $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ be nondecreasing. If there exists $\alpha > 0$ such that for all $t \geq 0$,

$$f(t) \leq \alpha \int_0^t f(s) ds + \psi(s),$$

then,

$$\sup_{s \in [0,t]} f(s) \leq e^{\alpha t} \psi(t).$$

**PROOF.** Soon

**Proposition 1.11** Assume that $b$ and $\sigma$ are Lipschitz. Then, for all $p \geq 2$, there exists a constant $c_p$ such that:

$$\left\| \sup_{t \in [0,T]} |X_t - \tilde{X}_t| \right\|_p \leq c_p \left( \frac{T}{n} + 1 \right).$$

**PROOF.** The proof is written in the case $p = 2$. Note that

$$X_t - \tilde{X}_t = \int_0^t (b(X_s) - b(\tilde{X}_s))ds + \int_0^t (\sigma(X_s) - \sigma(\tilde{X}_s))dW_s.$$

This yields, by Cauchy-Schwarz inequality,

$$\mathbb{E} \left[ \sup_{t \in [0,T]} |X_t - \tilde{X}_t| \right] \leq 2 \mathbb{E} \left[ \sup_{t \in [0,T]} \left( \int_0^t b(X_s) - b(\tilde{X}_s)ds \right)^2 + \int_0^t \sigma(X_s) - \sigma(\tilde{X}_s)dW_s \right] \leq 2T \mathbb{E} \left[ \int_0^T |b(X_s) - b(\tilde{X}_s)|^2 ds \right] + 2 \mathbb{E} \left[ \int_0^T (\sigma(X_s) - \sigma(\tilde{X}_s))^2 dW_s \right].$$

By assumption, there exist $c_b$ and $c_\sigma$ positive constants such that for all $(x,y) \in \mathbb{R}^2$, $|b(x) - b(y)| \leq c_b|x - y|$ and $|\sigma(x) - \sigma(y)| \leq c_\sigma|x - y|$. Then, by Doob’s inequality and Itô isometry,

$$\mathbb{E} \left[ \sup_{t \in [0,T]} |X_t - \tilde{X}_t| \right] \leq 2T c_b^2 \mathbb{E} \left[ \int_0^T |X_s - \tilde{X}_s|^2 ds \right] + 2 \mathbb{E} \left[ \int_0^T (\sigma(X_s) - \sigma(\tilde{X}_s))^2 dW_s \right] \leq 2T c_b^2 \mathbb{E} \left[ \int_0^T |X_s - \tilde{X}_s|^2 ds \right] + 8 \mathbb{E} \left[ \int_0^T (\sigma(X_s) - \sigma(\tilde{X}_s))^2 ds \right].$$

Therefore, there exists $C > 0$,

$$\mathbb{E} \left[ \sup_{t \in [0,T]} |X_t - \tilde{X}_t| \right] \leq C \mathbb{E} \left[ \int_0^T |X_s - \tilde{X}_s|^2 ds \right],$$

$$\leq C \left( \mathbb{E} \left[ \int_0^T |X_s - X_t|^2 ds \right] + \mathbb{E} \left[ \int_0^T |\tilde{X}_s - \tilde{X}_t|^2 ds \right] \right),$$

$$\leq C \int_0^T \mathbb{E} \left[ \sup_{0 \leq s \leq t} |X_s - X_t|^2 ds \right] + \mathbb{E} \left[ \int_0^T |\tilde{X}_s - \tilde{X}_t|^2 ds \right].$$

Then, the proof is concluded by Gronwald’s lemma. ■
1.3.4 Higher order discretization scheme: the Milstein scheme

It is possible to obtain better rates of convergence by approximating $\int_0^t b(X_s) \, ds$ et $\int_0^t \sigma(X_s) \, dW_s$ more precisely. For instance, the Milstein scheme is given by

$$X_{k+1} - X_k = b(X_k) \frac{T}{n} + \sigma(X_k) \sqrt{\frac{T}{n} \varepsilon_k} + \frac{\sigma^2(X_k) T}{2 n} (\varepsilon_k^2 - 1),$$

where the $(\varepsilon_k)_{0 \leq k \leq n-1}$ are i.i.d. with $\varepsilon_k \sim \mathcal{N}(0, 1)$.

1.4 Exercises

Brownian motion

Let $(W_t)_{t \geq 0}$ be a Brownian motion started at 0.

1. Show that the following processes are Brownian motions.
   a. $(W_{t+s} - W_s)_{t \geq 0}$ for all $t \geq 0$.
   b. $(\alpha W_{t+s})_{t \geq 0}$ pour tout $\alpha > 0$.

   For all $t \geq 0$, write $Z_t = W_{t+s} - W_s$. Then,
   $$\mathbb{E}[Z_t] = \mathbb{E}[W_{t+s} - W_s] = \mathbb{E}[W_{t+s}] - \mathbb{E}[W_s] = 0.$$

   On the other hand, $t \mapsto Z_t$ is continuous and $Z_0 = 0$. Finally, for all $0 \leq s \leq t$,
   $$\mathbb{E}[(W_{s+t} - W_s)(W_{t+s} - W_t)] = \mathbb{E}[W_{s+t}W_{t+s} + W_s^2 - W_{t+s}W_s - W_sW_{t+s}],$$
   $$= \mathbb{E}[W_{s+t}W_{t+s}] + \mathbb{E}[W_s^2] - \mathbb{E}[W_{s+t}W_s] - \mathbb{E}[W_sW_{t+s}],$$
   $$= s + t_0 + t_0 - t_0 - t_0,$$
   $$= s.$$  

By linearity, $(Z_t)_{t \geq 0}$ is a Gaussian process. Then, $(Z_t)_{t \geq 0}$ is a Brownian motion.

Write for all $t \geq 0$, $\tilde{Z}_t = \alpha W_{\frac{t}{\alpha}}$. By linearity, it is a centered Gaussian process, continuous and such that $\tilde{Z}_0 = 0$. Then. For all $(s, t) \in \mathbb{R}^2$ such that $s \leq t$,

$$\mathbb{E}[\tilde{Z}_s\tilde{Z}_t] = \mathbb{E}\left[ \left( \alpha W_{\frac{s}{\alpha}} \right) \left( \alpha W_{\frac{t}{\alpha}} \right) \right] = \alpha^2 e \left[ W_{\frac{s}{\alpha}} \alpha^2 \right] = \alpha^2 \frac{s}{\alpha^2} = s = s \wedge t,$$

which concludes the proof.

2. For all $0 \leq s \leq t$, compute $\mathbb{E}[W_s^2 W_t^2]$, $\mathbb{E}[W_s W_t]$ and $\mathbb{E}[(W_t - W_s)^2 Y]$ where $Y$ is a bounded random variable measurable with respect to $\sigma((W_u)_{0 \leq u \leq s})$.

Let $0 \leq s \leq t$. 


1.4 Exercises

Let \( \mathbb{E}[W_tW_s^2] = \mathbb{E}[W_t(W_t - W_s + W_s)^2] , \)
\[= \mathbb{E}[W_t(W_t - W_s)^2] + 2\mathbb{E}[W_t^2(W_t - W_s)] + \mathbb{E}[W_t^3] , \]
\[= \mathbb{E}[W_t] \mathbb{E}[(W_t - W_s)^2] + 2\mathbb{E}[W_t^2]\mathbb{E}[W_t - W_s] + \mathbb{E}[W_t^3] , \]
\[= 0 + 0 + 0 , \]
\[= 0 . \]

On the other hand,
\[\mathbb{E}[W_t|W_s] = \mathbb{E}[W_t - W_s + W_s|W_s] , \]
\[= \mathbb{E}[W_t - W_s|W_s] + \mathbb{E}[W_s|W_s] , \]
\[= \mathbb{E}[W_t - W_s|W_s] + W_s , \]
\[= W_s . \]

Finally,
\[\mathbb{E}[(W_t - W_s)^2Y] = \mathbb{E}[\mathbb{E}[(W_t - W_s)^2Y|\sigma\{W_u\}_{0\leq u\leq s})] , \]
\[= \mathbb{E}[Y\mathbb{E}[(W_t - W_s)^2|\sigma\{W_u\}_{0\leq u\leq s})] , \]
\[= \mathbb{E}[Y\mathbb{E}[(W_t - W_s)^2]] , \]
\[= \mathbb{E}[Y(t-s)] , \]
\[= (t-s)\mathbb{E}[Y] . \]

3. Let \( (B_t)_{t \geq 0} \) be a Brownian motion started at 0 independent of \( W \) and \( \rho \in (0,1) \). Show that \( (Z_t)_{t \geq 0} \) is a Brownian motion, where for all \( t \geq 0 \), \( Z_t = \rho W_t + \sqrt{1 - \rho^2}B_t . \)

For all \( (s,t) \in \mathbb{R}^2 \) such that \( s \leq t , \)
\[\mathbb{E}[Z_sZ_t] = \mathbb{E}[(\rho W_s + \sqrt{1 - \rho^2}B_s)(\rho W_t + \sqrt{1 - \rho^2}B_t)] , \]
\[= \rho^2 \mathbb{E}[W_sW_t] + (1 - \rho^2) \mathbb{E}[B_sB_t] + \rho \sqrt{1 - \rho^2} \mathbb{E}[W_sB_t + W_tB_s] , \]
\[= \rho^2 s + (1 - \rho^2)s + 0 , \]
\[= s . \]

**Brownian bridge**

Let \( (B_t)_{0 \leq t \leq 1} \) be a centered Gaussian process, centered, and such that for all \( 0 \leq s \leq t \leq 1 \), \( \text{Cov}(B_s, B_t) = \min\{s,t\} - st . \)

1. Prove that \( (\tilde{B}_t)_{0 \leq t \leq 1} \) has the same law as \( (B_t)_{0 \leq t \leq 1} \) where for all \( 0 \leq t \leq 1 \), \( \tilde{B}_t = B_{1-t} . \)
2. Let \( (W_t)_{t \leq 0} \) be a Brownian motion and define for all \( t \geq 0 \), \( \tilde{W}_t = W_{t} - W_1 . \) Prove that \( \tilde{W} \) has the same law as \( B \) and is independent of \( W_1 \).

**Reflection principle - simulation of a first passage time**

Let \( (W_t)_{t \leq 0} \) be a Brownian motion started at 0 and write \( S_t = \sup_{0 \leq s \leq t} W_s . \) For all \( a \leq b , b > 0 \), show that
\[P(S_t \geq b ; W_t \leq a) = P(W_t \geq 2b - a) . \]
Prove that $S_t$ has the same law as $|W_t|$ and provide, for all $x > 0$, the probability density function of

$$
\tau_x = \inf_{t \geq 0} \{W_t \geq x\}.
$$

**Simulation of the maximum of a Brownian motion**

Let $(W_t)_{t \leq 0}$ be a Brownian motion started at 0.

1. Show that for all $x, y$ in $\mathbb{R}$,

$$
\mathbb{P}\left(\max_{0 \leq s \leq t} W_s \geq y \bigg| W_t = x\right) = \exp\left(-\frac{2y(y-x)}{t}\right),
$$

when $y \geq \max(0, x)$.

2. Show that conditionally on $\{W_t = x\}$, $\max_{0 \leq s \leq t} W_s$ has the same distribution as

$$
Z = \frac{x + \sqrt{x^2 - 2t \log U}}{2},
$$

where $U$ is uniformly distributed on $(0, 1)$. 